GUIDE TO EVERYTHING QUADRATIC

FOR TEACHERS (and students too!)

JAMES TANTON

This material appears in the reference series *THINKING MATHEMATICS!*

Volume 1: ARITHMETIC = GATEWAY TO ALL
Volume 2: ADVANCED COUNTING AND ADVANCEDNUMBER SYSTEMS
Volume 4: FUNCTIONS AND THEIR GRAPHS

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THE ALGBERA OF QUADRATICS

THE ALGEBRA OF QUADRATICS

An expression of the form $ax^2 + bx + c$ with *x* the variable and *a*, *b*, and *c* fixed values (with $a \neq 0$) is called a *quadratic*. To *solve a quadratic equation* means to solve an equation that can be written in the form $ax^2 + bx + c = 0$.

WHY THE NAME QUADRATIC?

The prefix *quad-* means "four" and quadratic expressions are ones that involve powers of *x* up to the second power (not the fourth power). So why are quadratic equations associated with the number four?

Answer: These equations are intimately connected with problems about squares and quadrangles. (In fact, the word *quadratic* is derived from the Latin word *quadratus* for square.) Questions about quadrangles often lead to quadratic equations. For example, consider the problem:

A quadrangle has one side four units longer than the other. Its area is 60 square units. What are the dimensions of the quadrangle?

If we denote the length of one side of the quadrangle as *x* units, then the other must be $x + 4$ units in length. We must solve the equation: $x(x+4) = 60$, which is equivalent to solving the quadratic equation $x^2 + 4x - 60 = 0$.

Solving quadratic equations, even if not derived from a quadrangle problem, still involves the geometry of four-sided shapes. As we shall see, all such equations can be solved by a process of "completing the square."

Some quadratic equations are straightforward to solve, as the following series of examples shows:

EXAMPLE 1: Solve $x^2 = 100$

Answer: Easy. $x = 10$ or $x = -10$. □

EXAMPLE 2: Solve $(x+3)^2 = 100$

Answer: A tad more complicated but still easy.
We have:

$$x+3=10 \qquad \text{or} \qquad x+3=-10$$

yielding: $x=7$ or $x=-13$ □

EXAMPLE 3: Solve $(y-4)^2 = 25$

Answer: We have:

$$y-4=5 \qquad \text{or} \qquad y-4=-5$$

yielding:

$$y=9 \qquad \text{or} \qquad y=-1$$ □

EXAMPLE 4: Solve $4\left(p+2\right)^2 - 16 = 0$

Answer: Add 16:

$$4(p+2)^2 = 16$$

Divide by 4:

$$(p+2)^2 = 4$$

so

$$p+2=2 \qquad \text{or} \qquad p+2=-2$$

yielding:

$$p=0 \qquad \text{or} \qquad p=-4$$ □

EXAMPLE 5: Solve $\left(x+7\right)^2 + 9 = 0$

Answer: We have:

$$(x+7)^2 = -9$$

In the system of real numbers, it is impossible for a quantity squared to be negative. This equation has no solution

[**Comment:** If *complex numbers* are part of your studies (See *ADVANCED COUNTING AND ADVANCED NUMBER SYSTEMS*) we can continue and deduce that:

$$x+7=3i \qquad \text{or} \qquad x+7=-3i$$

yielding the solutions $x=-7+3i$ or $x=-7-3i$.] □

EXAMPLE 6: Solve $\left(x-1\right)^2 = 5$

Answer:
We have:

$$x-1=\sqrt{5} \quad \text{or} \quad x-1=-\sqrt{5}$$

so:

$$x=1+\sqrt{5} \quad \text{or} \quad x=1-\sqrt{5} \qquad \square$$

EXAMPLE 7: Solve $(x+3)^2 = 49$

Answer:
We have:

$$x+3=7 \quad \text{or} \quad x+3=-7$$

yielding:

$$x=4 \quad \text{or} \quad x=-10 \qquad \square$$

EXAMPLE 8: Solve $x^2+6x+9=49$

Answer: If one is extremely clever one might realize that this is a repeat of example 7: The quantity x^2+6x+9 happens to equals $(x+3)^2$. Thus this equation has the same solutions as before: $x=4$ and $x=-10$. How might one be clever to see that this is indeed a familiar example in disguise? $\square$

Example 8 reveals a general strategy for solving all quadratic equations:

If we can rewrite an equation in the form

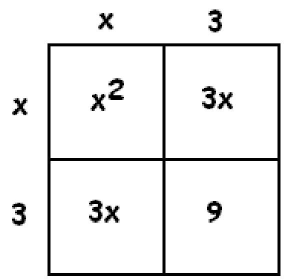
$$\left(x+A\right)^2 = B$$

then we have the solutions $x=-A+\sqrt{B}$ *or* $x=-A-\sqrt{B}$ *.*

We'll explore how to rewrite equations next.

COMPLETING THE SQUARE

An area model for multiplication provides a convenient interplay between arithmetic and geometry. For example, the quantity $(x+3)^2$ can be interpreted as the area of a square with two sides each of length $x+3$ inches. If we divide this square into four pieces, we see that $(x+3)^2 = x^2 + 6x + 9$.



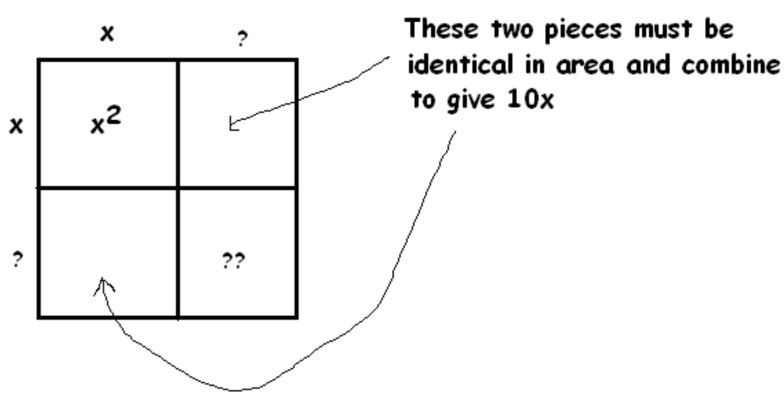
Notice that one piece of the square has area x^2, two pieces have equal area $3x$ and the fourth piece has area a specific numerical value.

EXAMPLE: Given:

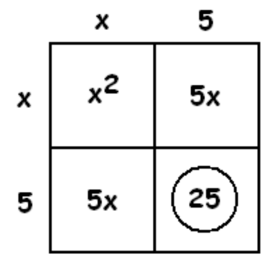
$$x^2 + 10x + ??$$

what number should one choose for "??" so that the picture associated with the expression is a perfect square?

Answer: Geometrically, the issue appears as follows:



Each of the two identical pieces must have area $5x$, and so the number in each of the places of the single question mark must be 5. This forces us to choose the number 25 in the place of the double question marks:



In choosing 25, we obtain $x^2 + 10x + 25 = (x+5)^2$, a perfect square $\square$

COMMENT: The process we followed is called *completing the square*. The quantity $x^2 + 10x$ becomes a complete square (literally!) once we add the value 25 to it.

EXERCISE: What number should one choose to make

$$x^2 + 8x + ??$$

a perfect square?

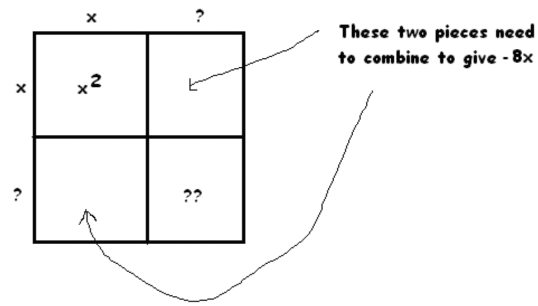
Let's allow for negative lengths and negative areas in a geometric model. (Although this has little meaning in terms of the geometry, the algebra implied by the admission of negative entities is still valid. See *ARITHMETIC: GATEWAY TO ALL* for a discussion of negative numbers and the geometric beliefs we still like to assign to them.)

EXAMPLE: What number should one choose to make

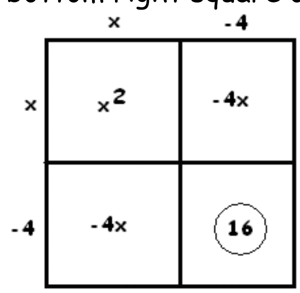
$$x^2 - 8x + ??$$

a perfect square?

Answer:



Select"-4" and "-4" making the bottom right square of area 16.



Thus choose "16" for the missing number.

$$x^2 - 8x + 16 = (x-4)^2 \qquad \square$$

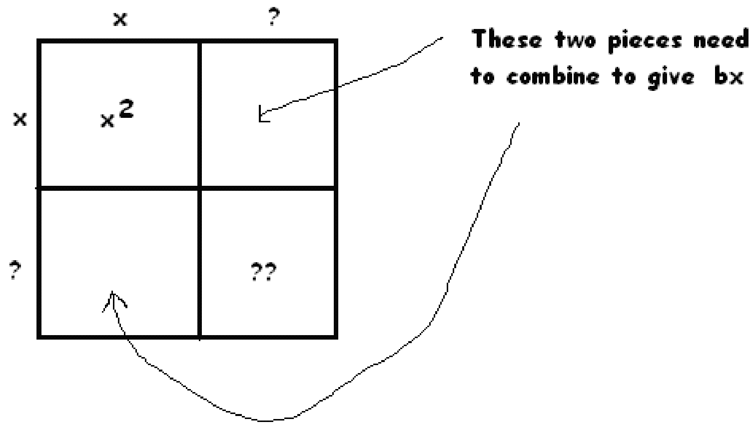
EXERCISE: Select a value for the missing term to make each of these a perfect square. Give the dimensions of the square.

a) $x^2 + 20x + ??$
b) $x^2 - 6x + ??$
c) $x^2 + 300x + ??$
d) $x^2 + 3x + ??$

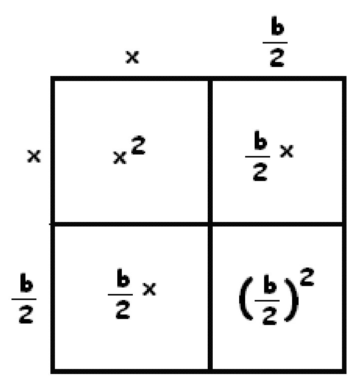
COMMENT: There is a general technique not worth memorizing. (If one ever needs this and can't recall the technique – just draw the box!)

To make $x^2+bx+??$ into a perfect square, always choose $\left(\frac{b}{2}\right)^2$ for the missing term.

Reason:



Select "$\frac{b}{2}$" and "$\frac{b}{2}$" making the bottom right square of area $\left(\frac{b}{2}\right)^2$.



Thus choose "half of *b* all squared" for the missing number.

$$x^2+bx+\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)^2 \qquad \square$$

EXAMPLE: Make $x^2 - 7x + ??$ into a perfect square.

Answer: Work with $\left(-\frac{7}{2}\right)^2$. We have $x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$

<u>CHECK</u>: $\left(x-\frac{7}{2}\right)\left(x-\frac{7}{2}\right) = x^2 - \frac{7}{2}x - \frac{7}{2}x + \frac{49}{4} = x^2 - 7x + \frac{49}{4}$ Yes! □

EXAMPLE: Make $x^2 + \sqrt{5}x + ??$ into a perfect square.

Answer: Work with $\left(\frac{\sqrt{5}}{2}\right)^2$. We have $x^2 + \sqrt{5}x + \frac{5}{4} = \left(x + \frac{\sqrt{5}}{2}\right)^2$

<u>CHECK</u>: $\left(x+\frac{\sqrt{5}}{2}\right)\left(x+\frac{\sqrt{5}}{2}\right) = x^2 + \frac{\sqrt{5}}{2}x + \frac{\sqrt{5}}{2}x + \frac{5}{4} = x^2 + \sqrt{5}x + \frac{5}{4}$ Yes! □

EXAMPLE: Make $x^2 + \frac{1}{2}x + ??$ into a perfect square.

Answer: Work with $\left(\frac{1}{4}\right)^2$. We have $x^2 + \frac{1}{2}x + \frac{1}{16} = \left(x + \frac{1}{4}\right)^2$

<u>CHECK</u>: $\left(x+\frac{1}{4}\right)\left(x+\frac{1}{4}\right) = x^2 + \frac{1}{4}x + \frac{1}{4}x + \frac{1}{16} = x^2 + \frac{1}{2}x + \frac{1}{16}$ Yes! □

EXERCISE: Make each of the following into a perfect square. Check that your answers are correct.

a) $x^2 + 5x + ??$

b) $x^2 - x + ??$

c) $x^2 + \frac{1}{3}x + ??$

d) $x^2 - 1.2x + ??$

e) $x^2 + \pi x + ??$

SOLVING QUADRATIC EQUATIONS: Continued

Let's continue our list of examples from page 6, using the technique of completing the square to help us out. Although many texts have students attempt factoring quadratics in order to solve them, this is a very artificial and contrived approach. (For example, who would think to guess the factors $1-\sqrt{5}$ and $1+\sqrt{5}$ when solving $x^2-2x-4=0$?) The method of completing the square, however, is a natural approach to quadratics (hence the name!) and absolutely guaranteed to work in all circumstances. The idea is to simply adjust the quadratic equation at hand so that a perfect square appears. A series of examples explains.

EXAMPLE 9: Solve $x^2+4x=21$

Answer: Which number makes $x^2+4x+??$ a perfect square? Four. (It never hurts to draw the box if it helps!)

	x	2
x	x^2	2x
2	2x	4

To solve

$$x^2+4x \quad = \quad 21$$

add four to both sides:

$$x^2+4x+4=25$$

This is:

$$(x+2)^2=25$$

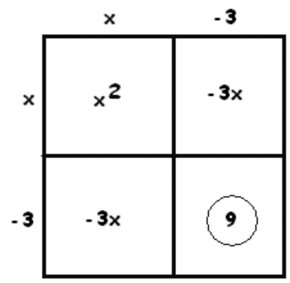
yielding

$$x+2=5 \qquad \text{or} \qquad x+2=-5$$

That is, $x=3$ or $x=-7$ □

EXAMPLE 10: Solve $x^2 - 6x \ = \ 55$

Answer: To obtain a perfect square add 9 to both sides.



$$x^2 - 6x + 9 = 64$$

$$(x-3)^2 = 64$$

$$x - 3 = 8 \qquad \text{or} \qquad x - 3 = -8$$

$$x = 11 \qquad \text{or} \qquad x = -5$$

□

EXAMPLE 11: Solve $x^2 + 8x + 10 \ = \ 30$

Answer: The "10" in the left side is the wrong number for a perfect square. Life would be easier if it were 16 instead. Let's make it 16!

Add 6 to both sides:

$$x^2 + 8x + 16 = 36$$

$$(x+4)^2 = 36$$

$$x + 4 = 6 \qquad \text{or} \qquad x + 4 = -6$$

$$x = 2 \qquad \text{or} \qquad x = -10$$

□

EXAMPLE 12: Solve $x^2 - 10x + 3 \ = \ 0$

Answer: The "3" in the left side is the wrong number for a perfect square. Let's make it 25.
Add 22 to both sides:

$$x^2 - 10x + 25 = 22$$

$$(x-5)^2 = 22$$

$$x - 5 = \sqrt{22} \qquad \text{or} \qquad x - 5 = -\sqrt{22}$$

$$x = 5 + \sqrt{22} \qquad \text{or} \qquad x = 5 - \sqrt{22}$$

□

EXAMPLE 13: Solve $w^2 + 90 = 22w - 31$

Answer: Let's bring all the terms containing a variable to the left side.

$$w^2 - 22w + 90 = -31$$

And adjust the number "90" by adding 31 throughout to complete the square.

$$w^2 - 22w + 121 = 0$$

Thus we have:

$$\left(w - 11\right)^2 = 0$$

giving

$$w - 11 = 0$$

That is:

$$w = 11.$$

□

COMMENT: Zero is the only number with a single square root. As this example shows, it is possible for a quadratic equation to have just one solution. (One normally expects two.)

COMMENT: LEADING COEFFICIENTS DIFFERENT FROM ONE

Consider a quadratic equation with the coefficient of the x^2-term different from 1, say:

$$3x^2 - 4x + 1 = 0$$

One can solve this by dividing through by 3 and completing the square for the equation $x^2 - \frac{4}{3}x + \frac{1}{3} = 0$. This will certainly work using the box method, which begins by noting that x^2 is x times x.

Alternatively … We could modify $3x^2$ to make it too a perfect square. Begin by multiplying the equation through by 3:

$$9x^2 - 12x + 3 = 0$$

and apply the box method to this equation. Note that one cell of the box has area $9x^2$ and so side-length $3x$:

	3x	
3x	9x^2	

We need two cells of the same area "adding" to $-12x$:

	3x	
3x	9x^2	- 6x
	- 6x	

	3x	- 2
3x	9x^2	- 6x
- 2	- 6x	4

and this shows that the constant term we need to complete the square is 4.

$$9x^2 - 12x + 3 = 0$$
$$9x^2 - 12x + 4 = 1$$
$$(3x-2)^2 = 1$$
$$3x - 2 = 1 \text{ or } -1$$
$$3x = 3 \text{ or } 1$$
$$x = 1 \text{ or } \frac{1}{3}$$

Done!

EXERCISE: Solve the following quadratics via this method:

a) $4x^2 - 12x + 5 = 0$
c) $2x^2 + 4x - 1 = 0$
d) $5x^2 - 2x - 3 = 0$
a) $2x^2 - 3x - 2 = 0$

A TRICK TO AVOID FRACTIONS

Solving $x^2 - 3x + 2 = 20$ requires working with fractions $-\frac{3}{2}$ and $\frac{9}{4}$. (Try it to see why.) The problem is the odd number in the middle of the quadratic.

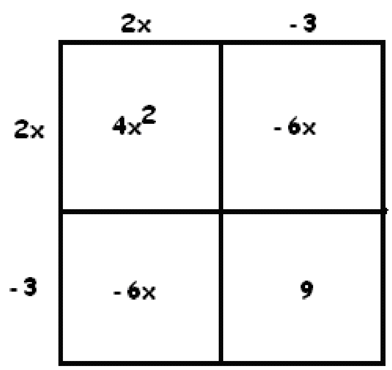
Here's a trick:

Make that middle number even by multiplying through by **FOUR**.

This gives:

$$4x^2 - 12x + 8 = 80$$

Why four? Doing so gives the term $4x^2$ which is $(2x)^2$. This allows us to complete the square:



Thus:

$$4x^2 - 12x + 8 = 80$$
$$4x^2 - 12x + 9 = 81$$
$$(2x-3)^2 = 81$$
$$2x - 3 = 9 \quad or \quad -9$$
$$2x = 12 \quad or \quad -6$$
$$x = 6 \ or \ -3$$

Exercise: Solve the following quadratics using this trick:

a) $x^2 - 7x + 2 = 0$

b) $2y^2 = 6y + 1$

THE ULTIMATE BOX METHOD

Let's now combine the two tricks we've just discussed to create the "ultimate" box method for all quadratics. Suppose we wish to solve an equation of the form:

$$ax^2 + bx + c = 0$$

where the number in front of x^2 is not one, and the number in front of x might be odd:

i) Multiply through by a and solve instead $a^2x^2 + abx + ac = 0$
ii) Also multiply through by 4 if the middle coefficient is odd and solve $4a^2x^2 + 4abx + 4ac = 0$
iii) Now use the box method on this equation

EXAMPLE: Solve $3x^2 + 7x + 4 = 0$.

Answer: Let's multiply through by 3:

$$9x^2 + 21x + 12 = 0$$

To deal with the odd middle term let's also multiply through by 4:

$$36x^2 + 84x + 48 = 0$$

Now we are set to go:

	6x	7
6x	36x²	42x
7	42x	49

$$36x^2 + 84x + 49 = 1$$
$$(6x+7)^2 = 1$$
$$6x + 7 = 1 \ \ or \ \ -1$$
$$6x = -6 \ \ or \ \ -8$$
$$x = -1 \ \ or \ \ -\frac{8}{7}$$

□

EXAMPLE: Solve $-2x^2 + 3x + 7 = 1$

Answer: Let's multiply through by -2 and also through by 4:

$$16x^2 - 24x - 56 = -8$$

Now the box method:

	4x	**-3**
4x	**16x²**	**-12x**
-3	**-12x**	**9**

$$16x^2 - 24x + 9 = -8 + 9 + 56$$
$$(4x-3)^2 = 57$$
$$4x - 3 = \pm\sqrt{57}$$
$$x = \frac{3 \pm \sqrt{57}}{4}$$

Done! □

EXERCISE: Solve the following quadratic via the ultimate box method:

a) $5x^2 - x - 18 = 0$
b) $3x^2 + x - 4 = 0$
c) $2x^2 - 3x = 5$
d) $10x^2 - 10x = 1$

COMMENT: Of course fractions and other types of numbers are unavoidable in solving quadratics with non-integer coefficients.

CHALLENGE: Solve $\sqrt{3}x^2 - \frac{1}{\sqrt{2}}x + \frac{1}{10} = 0$

Comment: The box method can be made to work!

******** OPTIONAL READING ********

ASIDE ON COMPLEX SOLUTIONS:
If one wishes to stay within the realm of real numbers, then it can be that a quadratic equation has no solutions. Within the realm of the complex numbers, however, square roots of negative quantities are permitted and all quadratic equations have solutions. (Usually two, sometimes just one.)

EXAMPLE 14: Solve $9+z^2=4z$

Answer: Let's bring all terms containing the variable to one side.

$$z^2-4z+9=0$$

Subtract five from both sides to obtain a perfect square:

$$z^2-4z+4=-5$$
$$(z-2)^2=-5$$

This has no solution in the realm of real numbers: no quantity squared is negative. In the realm of complex numbers, however, we can say:

$$z-2=i\sqrt{5} \text{ or } -i\sqrt{5}$$

yielding:

$$z=2+i\sqrt{5} \text{ or } 2-i\sqrt{5}$$ □

EXAMPLE 15: Solve $2x^2+2x+1=0$

Answer: Multiply through by 2:

$$4x^2+4x+2=0$$

Use the box method to see that the magic number we seek is "1":

$$4x^2+4x+1=-1$$
$$(2x+1)^2=-1$$

This has no solution within the reals. Within the system of complex numbers we can continue on:

$$2x+1=i \quad or \quad -i$$

yielding $x=\dfrac{-1+i}{2}$ or $\dfrac{-1-i}{2}$. □

THE GENERAL QUADRATIC FORMULA

Let's look at the general procedure we have developed for solving a quadratic equation of the form employing the two tricks that led to the "ultimate box method:

$$ax^2 + bx + c = 0$$

Since the leading term is ax^2, potentially not a square number, let's multiply through by a:

$$a^2x^2 + abx + ac = 0$$

The middle term abx might involve an odd coefficient. To avoid fractions, let's also multiply through by four:

$$4a^2x^2 + 4abx + 4ac = 0$$

Now apply the box method:

	2ax	
2ax	$4a^2x^2$	

	2ax	
2ax	$4a^2x^2$	2abx
	2abx	

	2ax	b
2ax	$4a^2x^2$	2abx
b	2abx	b^2

This shows that we need the "magic number" b^2 to complete the square. Let's subtract $4ac$:

$$4a^2x^2 + 4abx = -4ac$$

and add b^2:

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

Now we are set to go. (This looks worse than it actually is!)

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

$$(2ax + b)^2 = b^2 - 4ac$$

$$2ax + b = \sqrt{b^2 - 4ac} \; or \; -\sqrt{b^2 - 4ac}$$

Now add $-b$ throughout:

$$2ax = -b + \sqrt{b^2 - 4ac} \; or \; -b - \sqrt{b^2 - 4ac}$$

And divide through by $2a$:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

This statement can be combined into a single expression:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{\textbf{IF} \quad ax^2 + bx + c = 0 \quad \textbf{THEN} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

SOME LANGUAGE: THE DISCRIMINANT

A general quadratic $ax^2 + bx + c = 0$ has general solution:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Memorising this formula offers the speediest way to solve quadratics. (But if speed isn't your primary concern, then the box method we've described, that is, completing the square, will never let you down and will keep you close in touch with your understanding of why solving quadratics works the way it does!)

The quantity under the square root sign, $b^2 - 4ac$, is called the **discriminant** of the quadratic. Folk like to give this quantity a name because its sign determines the types of solutions one obtains:

- If $b^2 - 4ac$ is negative, then the quadratic has no real solutions. (One cannot compute the square root of a negative value.)

- If $b^2 - 4ac$ eauals 0, then the quadratic has precisely one solution. (Zero is the only number with precisely one square root.)

- If $b^2 - 4ac$ is positive, then the quadratic has two real solutions. (There are two square roots to a positive quantity.)

For example, without any effort we can see:

> $2w^2 - 3w + 4 = 0$ has no solutions because $b^2 - 4ac = 3^3 - 4 \cdot 2 \cdot 4 = -23$ is negative.
>
> $x^2 - 4x + 4 = 0$ has precisely one solution because $b^2 - 4ac = 16 - 16 = 0$.
>
> $3y^2 - y - 1$ has precisely two solutions because $b^2 - 4ac = 1 + 12 = 13$ is positive.

People also note that the two solutions to a quadratic $ax^2 + bx + c = 0$ can be written:

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

This shows that the two solutions lie at symmetrical positions about the value:

$$x = -\frac{b}{2a}$$

Some folk consider this important to hold in mind.

COMMENT: If you know the shape of a quadratic graph (see part II), then solutions to the quadratic equation $ax^2 + bx + c = 0$ correspond to where $y = ax^2 + bx + c$ crosses the *x*-axis and, because the graph is symmetrical, we have that $x = -\frac{b}{2a}$ is the location of the vertex of the graph. [But there are better ways to see this than from analyising the quadratic formula! See part II.]

SHOULD ONE MEMORISE THE QUADRATIC FORMULA?

If speed matters to you … maybe. If understanding has a higher priority in your mind, then the box method will not let you down … especially if you employ BOTH the tricks we used to derive the formula in the first place:

To solve $ax^2 + bx + c = 0$ multiply through by *a* to make ax^2 a square and also multiply through by 4 to avoid fractions.

EXERCISE:

a) A rectangle is twice as long as it is wide. Its area is 30 square inches. What are the length and width of the rectangle?

b) A rectangle is four inches longer than it is wide. Its area is 30 square inches. What are the length and width of the rectangle?

c) A rectangle is five inches longer than its width. Its area is 40 square inches. What are the dimensions of the rectangle?

EXERCISE: Solve the following quadratic equations:

a) $v^2 - 2v + 3 = 27$
b) $z^2 + 4z = 7$
c) $w^2 - 6w + 5 = 0$
d) $\alpha^2 - \alpha + 1 = \dfrac{7}{4}$

Also note that $x = \left(\sqrt{x}\right)^2$. Solve the following disguised quadratics.

e) $x - 6\sqrt{x} + 8 = 0$
f) $x - 2\sqrt{x} = -1$
g) $x + 2\sqrt{x} - 5 = 10$

WATCH OUT! Explain why only <u>one</u> answer is valid for g).

h) $3\beta - 2\sqrt{\beta} = 7$
i) $2u^4 + 8u^2 - 5 = 0$

EXERCISE:

a) Show that an expression of the form $a(x-h)^2 + k$ is quadratic.

b) If $y = a(x-h)^2 + k$ and *a* is positive, explain why the minimal value of *y* occurs when $x = h$. [Thus *y* takes all values *k* and higher.] How does this analysis change if *a* is negative?

THE CUBIC FORMULA

There is a reason why the "cubic formula" is not taught in schools. It's somewhat tricky. Let's develop the formula here through a series of exercises.

Consider an equation of the form: $x^3 + Ax^2 + Bx + C = 0$.

a) Put $x = z - \frac{A}{3}$ into this equation to show that it becomes an equation of the form: $z^3 = Dz + E$ for some new constants D and E.

Thus, when solving cubic equations, we can just as well assume that no x^2 term appears. This trick was known by mathematicians of the 16th century. Then, Italian mathematician Girolamo Cardano (1501-1576) came up with the following truly inspired series of steps for going further.

Instead of calling the constants D and E, he suggested calling them $3p$ and $2q$. This means we need to solve the equation:

$$z^3 = 3pz + 2q$$

b) Show that if s and t are two numbers that satisfy $st = p$ and $s^3 + t^3 = 2q$, then $z = s + t$ will be a solution to the cubic.

So our job now is to find two numbers *s* and *t* satisfying $st = p$ and $s^3 + t^3 = 2q$.

c) Eliminate t between these two equations to obtain a quadratic formula in s^3.

d) Without doing any extra work, write down the quadratic equation we would have for t^3 if, instead, we eliminated s between the two equations.

e) Show that a solution to the cubic equation $z^3 = 3pz + 2q$ is:

$$z = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

[Be careful about the choices of signs here. Recall we must have $s^3 + t^3 = 2q$.]

THAT'S IT. That's the cubic formula (once you untangle the meaning of *p* and *q*).

For practice:

f) Solve $z^3 = 3z + 2$.

g) Solve $z^3 = 6z + 6$.

h) Solve $x^3 + 3x^2 - 3x - 11 = 0$.

i) Solve $x^3 = 6x - 4$

COMMENT: This final example is interesting. Graphically, the curve $y = x^3$ is the standard cubic curve and $y = 6x - 4$ is a straight-line graph. It is clear from the graphs that these two curves must intersect and produce a real solution. Is your answer to h) a real number? Is it a real number in disguise?
(HINT: Evaluate $(1+i)^3$ and $(1-i)^3$.)

This very issue led Italian mathematician Rafael Bombelli (1526-1572/3) to accept complex numbers (see volume 2) as "valid" and useful entities for obtaining real solutions to problems.

INTERNET EXERCISE: Is there a formula for solving quartic equations?

INTERNET RESEARCH: Is there a formula for solving quintic equations and beyond?

[There is a rich story in the history of mathematics here.]

PART II:
GRAPHING QUADRATICS

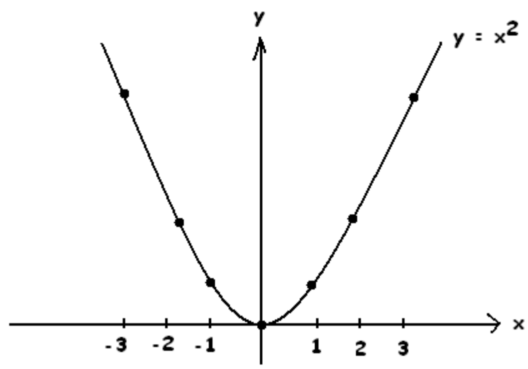
GRAPH OF THE SQUARING FUNCTION

Consider the squaring function $y = x^2$ which takes an input x and squares it to produce the output x^2. For example, if:

If $x = 1$, then $y = 1^2 = 1$.
If $x = 5$, then $y = 5^2 = 25$.
If $x = -5$, then $y = (-5)^2 = 25$.
If $x = 20$, then $y = 20^2 = 400$
If $x = 0$, then $y = 0^2 = 0$.

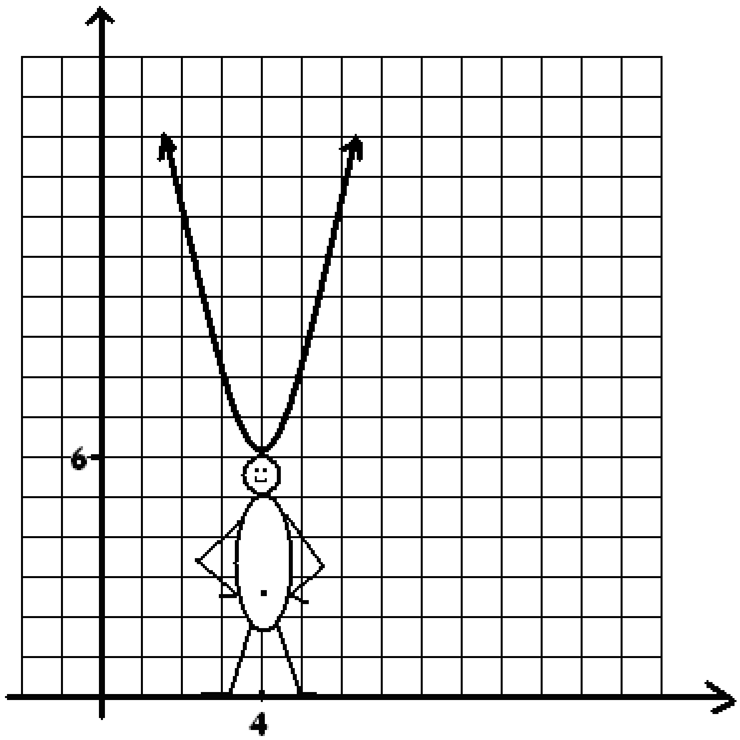
and so on. If we draw a table and plot points, we see that the graph of $y = x^2$ is an upward facing U-shaped curve.

x	0	1	2	3	-1	-2	-3
y	0	1	4	9	1	4	9



Now here is a challenge question.

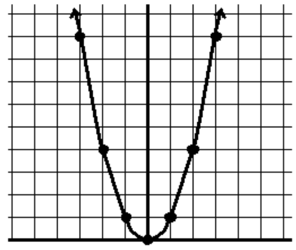
> *I am six units tall and am standing at the position* $x = 4$ *on the horizontal axis. Is it possible to write down a formula for a function whose graph is the same U-shaped curve as for* $y = x^2$ *but positioned to balance on my head?*



Perhaps try this before reading on. Play with some possible formulae. Plug in some sample points and table values. Test whether your ideas offer any hints as to a solution to this challenge.

FROM $y = x^2$ to $y = (x-k)^2$

Here is the graph of $y = x^2$ again:



It has a table of values:

x	-3	-2	-1	0	1	2	3	4	5
y=x^2	9	4	1	0	1	4	9	16	25

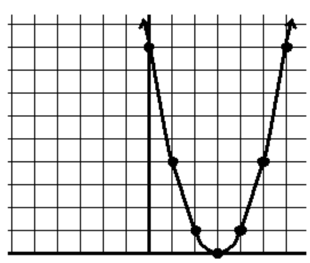
One thing to notice is that this graph has a "dip" at $x = 0$.

Now consider the function $y = (x-3)^2$.

Notice that when we put $x = 3$ into this formula we obtain the value 0^2. That is, the number 3 is "behaving" just like $x = 0$ was for the original function.

In $y = (x-3)^2$ we have that 3 is the "new zero" for the x-values.

So whatever the original function was doing at $x = 0$, it is now doing it at $x = 3$. The original function $y = x^2$ "dips" at $x = 0$ so the graph of the function $y = (x-3)^2$ dips at $x = 3$.

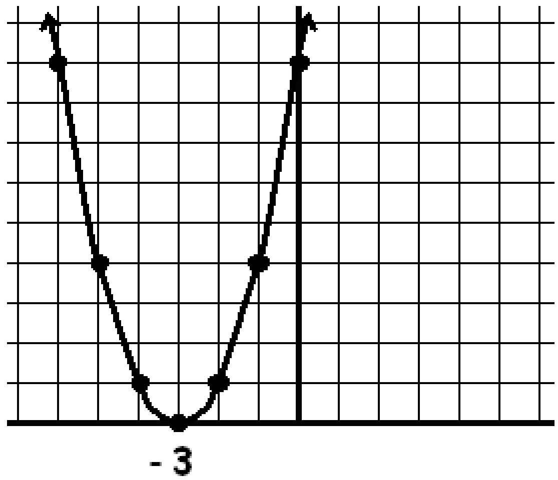


The entire graph has been shifted horizontally – and one can check this by drawing a table of values.

x	-3	-2	-1	0	1	2	3	4	5
y = (x- 3)^2	36	25	16	9	4	1	0	1	4

EXAMPLE: Sketch a graph of $y=(x+3)^2$.

Answer: What value of *x* behaves like zero for $y=(x+3)^2$? Answer: $x=-3$ does! So $y=(x+3)^2$ looks like $y=x^2$ but with $x=-3$ the new zero.



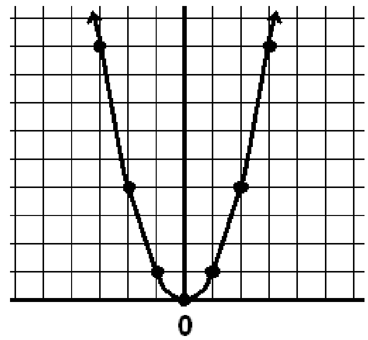
EXERCISE:

a) Sketch a graph of $y=(x-4)^2$

b) Sketch a graph of $y=\left(x+\frac{1}{2}\right)^2$

FROM $y = x^2$ to $y = x^2 + b$

Here's the function $y = x^2$ again:



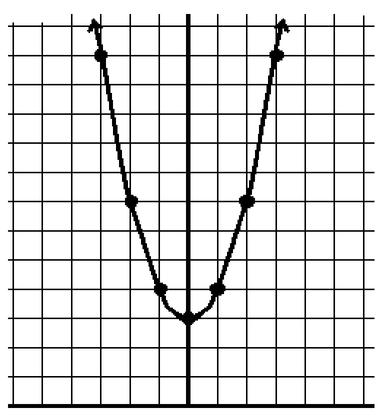
x	-3	-2	-1	0	1	2	3	4	5
$y=x^2$	9	4	1	0	1	4	9	16	25

How would the graph of $y = x^2 + 3$ appear?

Notice that this new function is adds three units to each output:

x	-3	-2	-1	0	1	2	3	4	5
$y = x^2 + 3$	12	7	4	3	4	7	12	19	28

This has the effect of raising the entire graphs three units in the vertical direction:

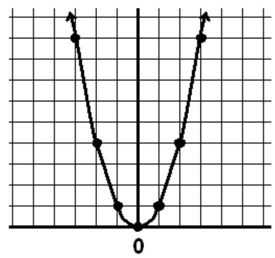


The graph of the function $y = x^2 - 5$ would be the same graph shifted downwards 5 units, and the graph of $y = x^2 + \sqrt{3}$ would the graph shifted upwards $\sqrt{3}$ units.

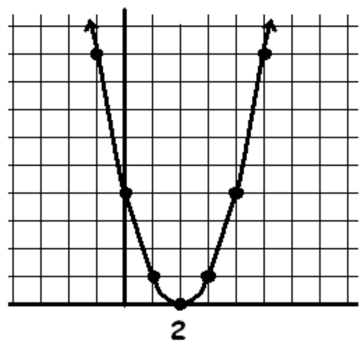
EXERCISE: a) Sketch $y = x^2 - 5$ b) Sketch $y = x^2 + \frac{1}{2}$

FROM $y = x^2$ TO $y = (x-k)^2$ TO $y = (x-k)^2 + b$

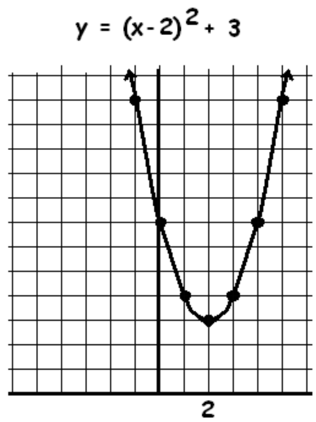
Here's the graph of $y = x^2$:



and here is the graph of $y = (x-2)^2$. Here "2" is acting as the new zero for the *x*-values, so the dip that was occurring at zero is now occurring at 2:



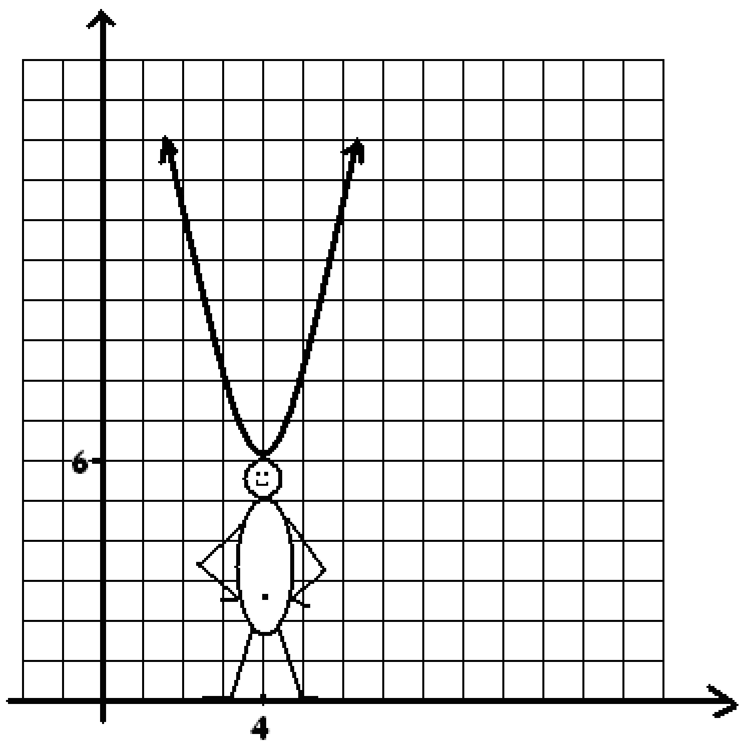
Now, consider $y = (x-2)^2 + 3$. This is the previous graph shifted upwards three units:



Question: Here we went from $y = x^2$ to $y = (x-2)^2$ to $y = (x-2)^2 + 3$ drawing the three graphs above along the way. Is it possible to think, instead, of the sequence "from $y = x^2$ to $y = x^2 + 3$ to $y = (x-2)^2 + 3$"?

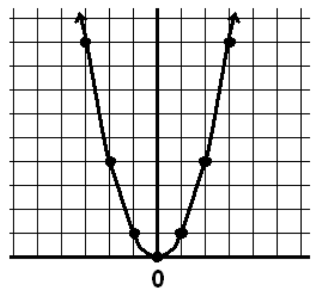
EXERCISE:
a) Sketch $y = (x-4)^2 - 1$
b) Sketch $y = (x+4)^2 + 2$
c) Sketch $y = (x+1)^2 - 2$

EXERCISE: I am six units tall and am standing at the position $x = 4$ on the horizontal axis. Write down the formula of a U-shaped graph that sits balanced on my head.



FROM $y = x^2$ TO $y = ax^2$

Here is the graph $y = x^2$ again:

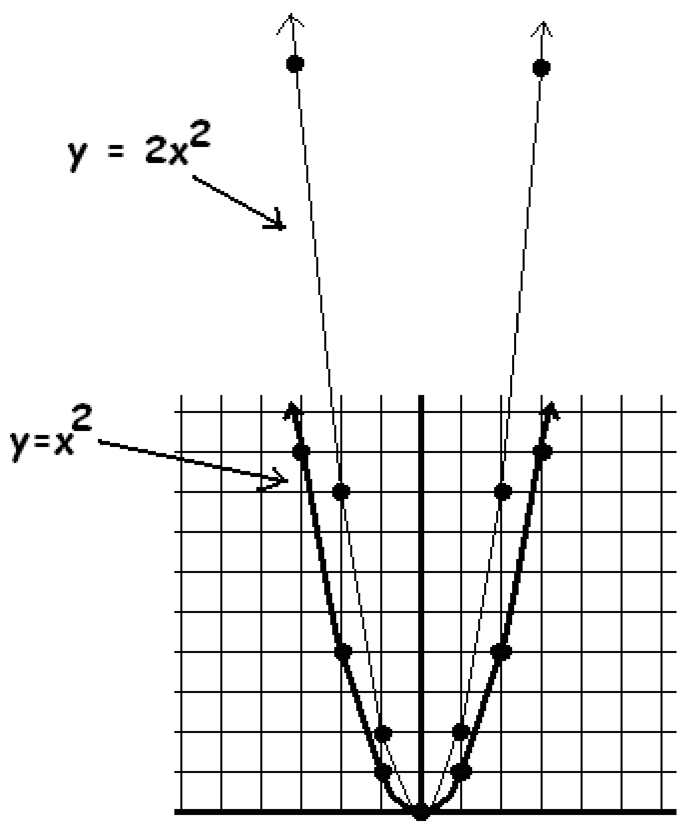


x	-3	-2	-1	0	1	2	3	4	5
$y=x^2$	9	4	1	0	1	4	9	16	25

What can we say about the graph of $y = 2x^2$? Certainly all the outputs are doubled:

x	-3	-2	-1	0	1	2	3	4	5
$y=2x^2$	18	8	2	0	2	8	18	32	50

This creates a "steeper" U-shaped graph:



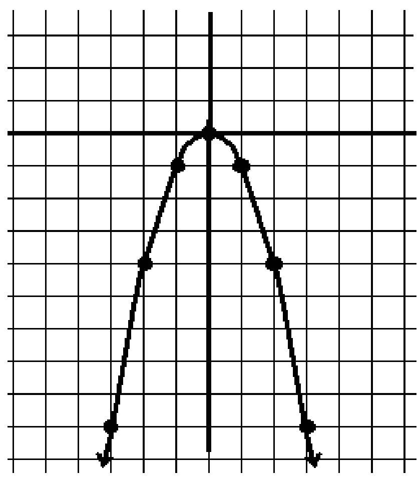
EXERCISE:

a) Draw a table of values for $y=3x^2$ and sketch its graph.

b) Which graph is "steeper," that for $y=100x^2$ or that for $y=200x^2$?

Consider $y=-x^2$. It's outputs are the negative of the outputs for $y=x^2$:

x	-3	-2	-1	0	1	2	3	4	5
$y=x^2$	9	4	1	0	1	4	9	16	25
$y=-x^2$	-9	-4	-1	0	-1	-4	-9	-16	-25



EXERCISE: Describe the graph of $y=-2x^2$.

EXERCISE:

a) Quentin says that the graph of $y=\frac{1}{10}x^2$ will be a very "broad" upward facing U-shaped graph. Is he right? Explain.

b) Describe the graph of $y=-\frac{1}{1000000}x^2$.

PUTTING IT ALL TOGETHER

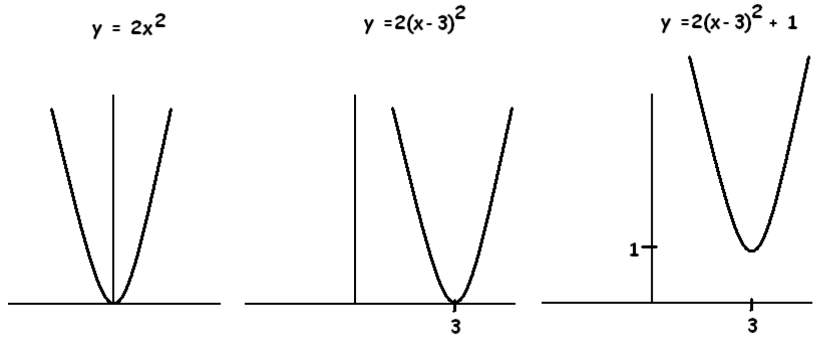
EXAMPLE: Analyse and quickly sketch $y = 2(x-3)^2 + 1$.

Answer: Now $y = 2(x-3)^2 + 1$ is essentially the graph $y = 2x^2 + 1$ with $x = 3$ made the new zero.

And $y = 2x^2 + 1$ is essentially $y = 2x^2$ with all outputs shifted upwards one unit.

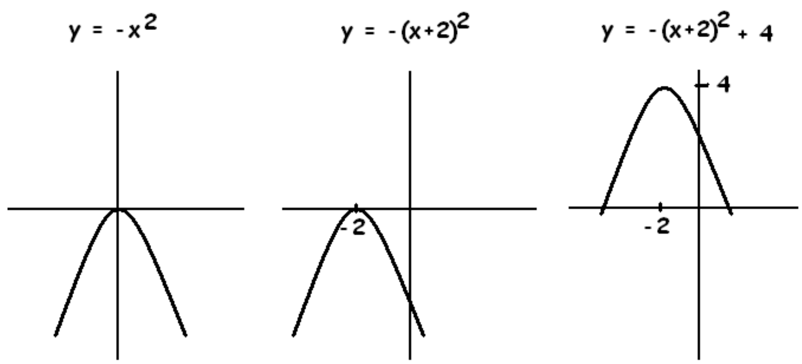
And $y = 2x^2$ is a steep upward facing U-shaped graph.

Putting it all together gives:



EXAMPLE: Analyse and quickly sketch $y = -(x+2)^2 + 4$.

Answer: $y = -(x+2)^2 + 4$ is essentially $y = -x^2$ with $x = -2$ made the new zero and shifted upwards four units.



IN SUMMARY …

We have just shown that the graph of any function of the form

$$y = a(x-k)^2 + b$$

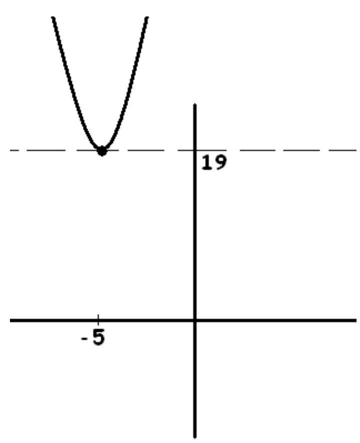
is a symmetrical U-shaped graph.

> **If *a* is positive** (as for $y = 700(x-56)^2 + 19$, for example), **then the U-shape faces upwards.**
>
> **If *a* is negative** (as for $y = -89(x-17)^2 + 92$, for example), **then the U-shape faces downwards.**

The graph is symmetrical about the "dip" which occurs at the new zero, that is, at $x = k$.

EXAMPLE: Describe the graph of $y = 4(x+5)^2 + 19$

Answer: This is an upward facing U-shaped graph.
Its dip occurs at $x = -5$. It is a fairly steep U-shaped curve.

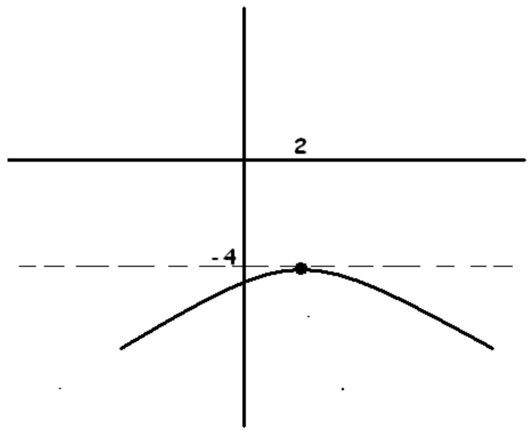


We can go further … Since any quantity squared, such as $(x+5)^2$, is sure to be greater than or equal to zero, this means that $y = 4(x+5)^2 + 19$ is always sure to have value 19 and higher. Thus our U-shaped graph is sits above $y = 19$ and in fact equals 19 when $x = -5$. This is clear from the picture of the graph too.

If we want we can add that the *y*-intercept occurs when $x = 0$, giving $y = 4 \cdot 25 + 19 = 119$.

EXAMPLE: Describe the graph of $y = -\frac{1}{3}(x-2)^2 - 4$

Answer: This is a downward facing U-shaped graph, symmetrical about the "new zero" of $x = 2$ shifted downward four units. It is a fairly broad U-shape.



If we like we can add that the *y*-intercept occurs when $x = 0$, giving $y = -\frac{1}{3} \cdot 4 - 4 = -\frac{16}{3}$.

Some folk like to specifically note:

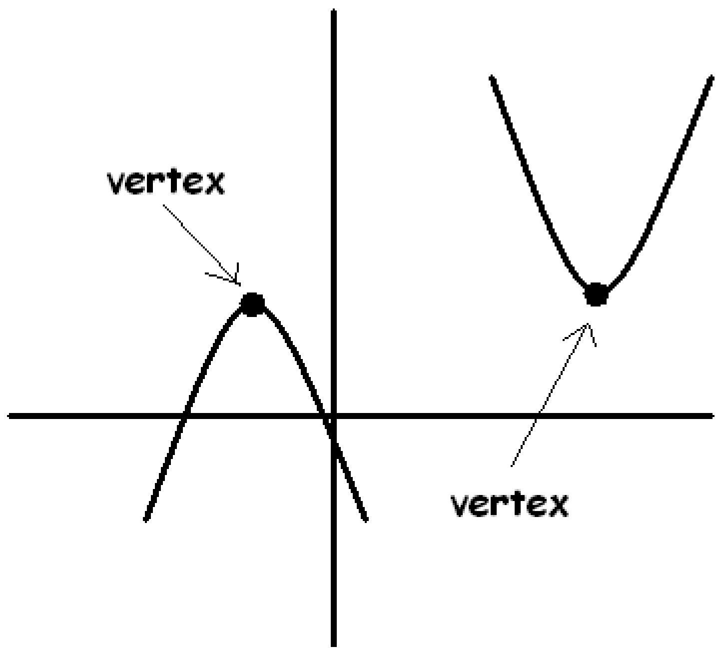
$y = a(x-k)^2 + b$ **takes values *b* and higher (if *a* is positive).**

$y = a(x-k)^2 + b$ **takes values *b* and lower (if *a* is negative).**

SOME LANGUAGE:

The U-shaped graphs that arise from the curves $y = a(x-k)^2 + b$ are called parabolas (because they turn out to precisely the curves ancient Greek scholars studied with regard to slices of cones, which they called parabolas).

The place where the parabola dips down to its lowest point, if it is upward facing, or "dips up" to its highest point if it is downward facing is called the vertex of the parabola.



We have seen that if $y = a(x-k)^2 + b$ then the vertex occurs at the "new zero" which is $x = k$. The *y*-value here is $y = a(k-k)^2 + b = 0 + b = b$. Thus the coordinates of the vertex are: (k,b).

[But don't memorize this! Just understand what you see as you do any specific example!]

GRAPHING OTHER QUADRATICS:

We'll show that equations of each of the form:

$$y = ax^2 + bx + c$$

are disguised versions of $y = a(x-k)^2 + b$ and so are also U-shaped curves.

GRAPHS OF $y = ax^2 + bx + c$

First the hard way …

EXAMPLE: Describe the graph of $y = x^2 + 2x + 9$

HARD Answer: Look at the portion $x^2 + 2x$. If we wish to complete the square on this piece (See PART I), we need a "+1" along with it. Do we have this? No quite, but let's make it happen:

$$y = x^2 + 2x + 9$$
$$y - 8 = x^2 + 2x + 1$$
$$y - 8 = (x+1)^2$$
$$y = (x+1)^2 + 8$$

Thus, $y = x^2 + 2x + 9$ is an upward facing parabola, symmetrical about $x = -1$, adopting values 8 and higher. The vertex occurs at $(-1, 8)$.

EXAMPLE: Describe the graph of $y = 3x^2 + 12x - 1$

HARD Answer: Let's complete the square on the piece $3x^2 + 12x$. As we saw in part I, it would be easier to complete the square by first multiplying this by 3. Let's do it!

$$3y = 9x^2 + 36x - 3$$

The box method shows that we want "+36" rather than " -3."

	3x	6
3x	9x^2	18x
6	18x	36

Let's add 39:

$$3y + 39 = 9x^2 + 36x + 36$$
$$3y + 39 = (3x + 6)^2$$
$$3y = (3x + 6)^2 - 39$$
$$y = \frac{1}{3}(3x + 6)^2 - 13$$

This is an upward facing parabola with $x = -2$ the new zero for the *x*-values. (Why?). The vertex is at $(-2, -13)$.

EXERCISE: Show that $y = \frac{1}{3}(3x + 6)^2 - 13$ can be rewritten as $y = 3(x + 2)^2 - 13$. Now it is very clear that $x = -2$ is indeed the new zero for the *x*-values.

$\boxed{\textbf{EXAMPLE: } \text{Describe the graph of } y = 4 - 16x - 2x^2}$

HARD Answer: Following the methods of part I, let's multiply through by -2:

$$-2y = 4x^2 + 32x - 8$$

Drawing the box shows that we want a constant term of +64:

$$-2y + 72 = 4x^2 + 32x + 64$$
$$-2y + 72 = (2x+8)^2$$
$$y = -\frac{1}{2}(2x+8)^2 + 36$$

This is a downward facing parabola, symmetrical about $x = -4$, taking values 36 and lower. The vertex is $(-4, 36)$.

CHECKING THE THEORY …
These examples show indicate that every quadratic $y = ax^2 + bx + c$ really is a transformed version of $y = x^2$, and so is either an upward facing U or a downward facing U.

We'll make use of this fact in a moment to develop a ridiculously easy way to graph quadratics . But in the meantime, here is an abstract proof in all its glory that shows this claim as true.

CLAIM: $y = ax^2 + bx + c$ is a transformed version of $y = x^2$.

REASON (WARNING: TOUGH OPTIONAL READING!)
We start with $y = ax^2 + bx + c$.

Let's follow the method of part I by multiplying through by a and then through by 4:

$$4ay = 4a^2x^2 + 4abx + 4ac$$

Now consider the box that goes with this.

	2ax	b
2ax	$4a^2x^2$	2abx
b	2abx	b^2

This tells us how to adjust the constant term:

$$4ay - 4ac = 4a^2x^2 + 4abx$$

$$4ay - 4ac + b^2 = 4a^2x^2 + 4abx + b^2$$

$$4ay - 4ac + b^2 = (2ax + b)^2$$

Solving for y produces:

$$y = \frac{1}{4a}(2ax + b)^2 + \frac{4ac - b^2}{4a}$$

$$y = \frac{1}{4a}\left(2a\left(x + \frac{b}{2a}\right)\right)^2 + stuff$$

$$y = \frac{1}{4a}4a^2\left(x + \frac{b}{2a}\right)^2 + stuff$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + stuff$$

And despite the visual horror of this, we see that it is just the graph $y = x^2$ transformed with some constants – just as we claimed!

Thus $y = ax^2 + bx + c$ is really of the form $y = a(x+h)^2 + n$ in disguise and so is a parabola. It is upward facing if a is positive, downward facing if a is negative.

All we need to take from this that $y = ax^2 + bx + c$ is a parabola, upward or downward facing depending on the sign of a.

NOW FOR …

THE EXTRAORDINARILY QUICK WAY TO GRAPH QUADRATICS

As an example, consider $y = x^2 + 4x + 5$.

We know that this is going to be an upward-facing U-shaped graph.

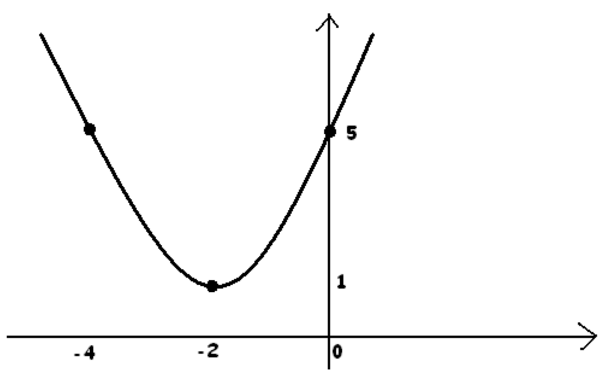
Pull out a common factor of x from the first two terms and write the expression as:

$$y = x(x+4) + 5.$$

This shows that $x = 0$ and $x = -4$ are interesting x-values that yield the same output of 5. Thus we have two symmetrical points on the parabola: $(-4,5)$ and $(0,5)$.

Since we know that the parabola is symmetrical, the vertex of the parabola must be half-way between these two x-values that yield the same output, namely, at $x = -2$. Substituting in gives the vertex at $(-2,1)$.

These three points allow us to sketch the quadratic.



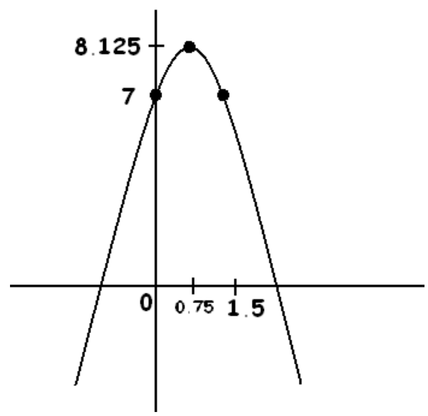
EXAMPLE: Make a quick sketch of $y = -2x^2 + 3x + 7$.
What are its x-and y-intercepts?

Answer: This is a downward facing parabola. We have:

$$y = -2x^2 + 3x + 7 = x(-2x+3) + 7$$

and this parabola has the value 7 at both $x=0$ and $x=\frac{3}{2}$. Because the graph is symmetrical, the vertex must be halfway between these values, at $x=\frac{3}{4}$. At this value, $y=\frac{3}{4}\left(-2\cdot\frac{3}{4}+3\right)+7=\frac{9}{8}+7=8\frac{1}{8}$.

The graph appears:



TWO x-VALUES THAT GIVE THE SAME OUTPUT FOR A QUADRATIC REPRESENT TWO SYMMETRICAL POINTS ON THE SYMMETRICAL CURVE.

THE VERTEX OF THE PARABOLA MUST LIE HALF-WAY BETWEEN THESE x-VALUES.

QUADRATICS OF THE FORM $y = a(x-p)(x-q)$

Consider, for example, the formula:

$$y = 2(x-3)(x+8)$$

If we expand brackets we see that this can be rewritten:

$$y = 2x^2 + 10x - 48$$

and so the graph of this function is again an (upward facing) parabola.

In the same way, expanding brackets shows that $y = -3(x+4)(x-199)$ is a downward facing parabola. (CHECK THIS!)

In general:

> $y = a(x-p)(x-q)$ **is a parabola; upward facing parabola if *a* is positive, downward facing if *a* is negative.**

Quadratics that happen to be in this factored form have the nice property that one can easily read off its *x*-intercepts. THEY HAVE TWO OBVIOUS "INTERESTING x-VALUES"

EXAMPLE: Where does $y = 2(x-3)(x+8)$ cross the *x*-axis? What is the *x*-value of its vertex? Briefly describe the graph of this function.

Answer: Can you see that $y = 0$ for $x = 3$ and for $x = -8$? Thus the graph of this function crosses the *x*-axis at these two values.

Because the graph is symmetrical (an upward facing parabola), the vertex occurs halfway between these two zeros. That is, the vertex occurs at $x = \frac{(-8)+3}{2} = -\frac{5}{2}$.

Here the *y*-value of the graph is $y = 2\left(-\frac{11}{2}\right)\left(\frac{11}{2}\right) = -\frac{121}{2} = -60\frac{1}{2}$.

Just for kicks, the *y*-intercept is (put $x = 0$): $y = 2(-3)(8) = -48$.

The graph is thus:

An upward facing parabola with vertex $\left(-2\frac{1}{2},-60\frac{1}{2}\right)$, crossing the *x*-axis at $x=-8$ and $x=3$. The *y*-intercept is $y=-48$

EXERCISE: Quickly sketch the following quadratics. (The key is to look for interesting x-values- that is, ones that give symmetrical locations on the symmetrical curve.)

a) $y=6x+x^2-1$
b) $y=4x^2+20x+80$
c) $y=6-3x^2-30x$
d) $y=2(x-5)(x-11)$
e) $y=-3(x+4)(x-4)$
f) $y=-x(x+6)$

EXERCISE: Here are three quadratics:

(A) $y=3(x-3)(x+5)$
(B) $y=2x^2+6x+8$
(C) $y=2(x-4)^2+7$

Here are four questions:

i) What is the smallest output the quadratic produces?

ii) Where does the quadratic cross the *x*-axis?

iii) Where does the quadratic cross the *y*-axis?

iv) What are the coordinates of the vertex of the quadratic?

For which of the three quadratics is it easiest to answer question i)?
For which of the three quadratics is it easiest to answer question ii)?
For which of the three quadratics is it easiest to answer question iii)?
For which of the three quadratics is it easiest to answer question iv) ?

EXERCISE:

a) Attempt to solve the equation $x^2+10x+30=0$. What do you deduce?

b) Sketch the graph of $y=x^2+10x+30$.

c) Use the graph to explain geometrically why there was is no solution to $x^2+10x+30=0$.

d) According to the graph, should there be solutions to $x^2+10x+30=11$? If so, find them.

e) Find a value *b* so that the equation $x^2+10x+30=b$ has exactly one solution.

EXERCISE:

a) Show that $y=2(x-1)(x+1)+(x-3)(x+4)+x(x-2)$ is a quadratic in disguise.

b) Sketch the quadratic.

EXERCISE: Consider the quadratic $y=ax^2+bx+c$. Rewrite this as:

$$y=x(ax+b)+c$$

a) The *x*-coordinate of parabola's vertex lies between which two values?

b) Explain why the vertex of the parabola occurs at $x=-\frac{b}{2a}$.

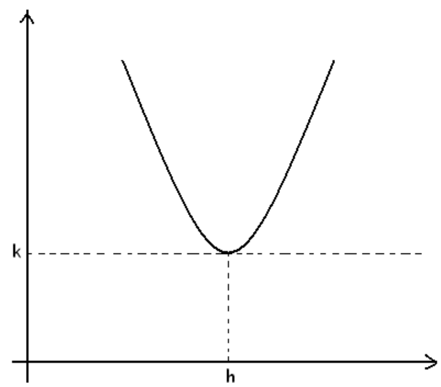
COMMENT: Many teachers make their students memorise this result. For example, given $y=3x^2+4x+8$, say, they like students to be able to say that its vertex lies at $x=-\frac{b}{2a}=-\frac{4}{2\cdot 3}=-\frac{2}{3}$. If speed is important to you, then great! If not, there is nothing wrong with writing $y=x(3x+4)+8$ and saying that the vertex is halfway between $x=0$ and $x=-\frac{4}{3}$.

OPTIONAL ASIDE: For those that know complex numbers …

GRAPHING COMPLEX SOLUTIONS

As we have seen, yny quadratic $y = ax^2 + bx + c$ can be written in the form $y = a\left(x-h\right)^2 + k$, and so is a transformed version of $y = x^2$, a U-shaped graph.

If *a* is positive, we see that the function $y = a\left(x-h\right)^2 + k$ takes all values *k* and higher, with the point (h,k) being the vertex of the parabola.



If, along with *a* being positive, *k* is positive, then graph fails to cross the *x*-axis, meaning that the equation $a\left(x-h\right)^2 + k = 0$ has no real solutions, only complex solutions:

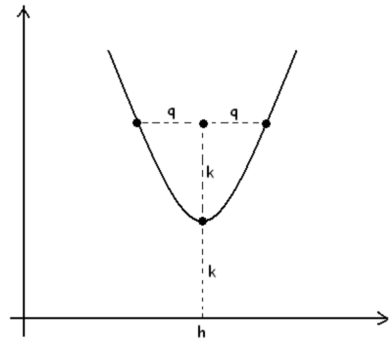
$$a\left(x-h\right)^2 + k = 0$$

$$\left(x-h\right)^2 = -\frac{k}{a}$$

$$x - h = \pm i\sqrt{\frac{k}{a}}$$

$$x = h \pm i\sqrt{\frac{k}{a}}$$

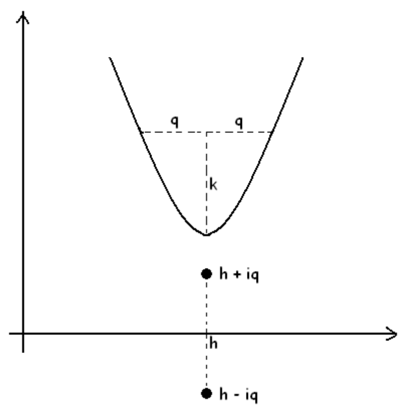
Surprisingly, there an easy way to locate those solutions on the graph!



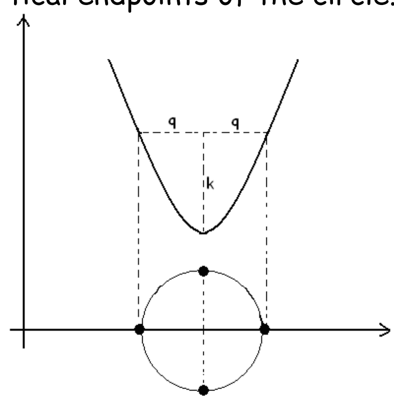
With a ruler, measure a vertical line twice the height of the vertex. Then measure the horizontal distance *q*, either left or right, to the parabola. We have $q = \sqrt{\frac{k}{a}}$.

(CHECK THIS by showing that $a(x-h)^2 + k = 2k$ has solutions $x = h \pm \sqrt{\frac{k}{a}}$.)

If we think of the plane of the graph as the complex plane, then the roots of the quadratic lie at the positions shown:



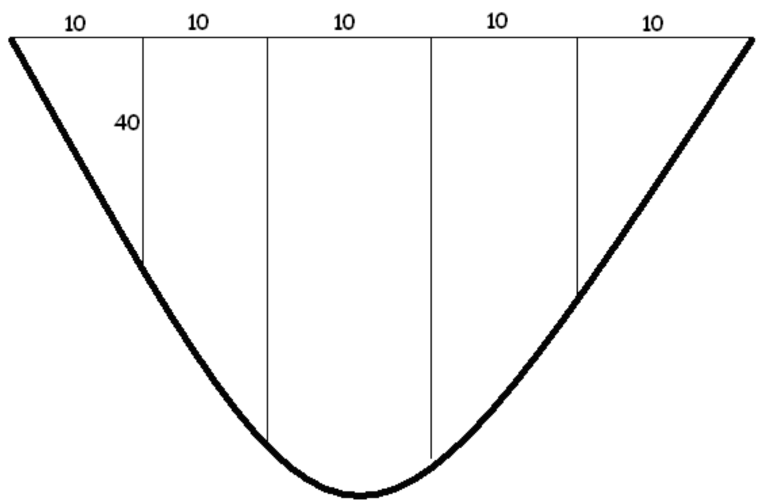
A swift way to locate these points is to draw a circle with its two *x*-intercepts as the endpoints of a diameter as shown. This gives a circle of radius *q* and the two complex roots lie at the vertical endpoints of the circle.



Here is a typical question on quadratics that hope-to-be mathematics teachers must answer on a state licensure exam. How would you fare with this?

TEACHER LICENSURE TYPE-QUESTION
Geologists suspect that the cross-section shape of a newly discovered impact crater in the Australian outback can be well approximated by a parabola. The crater is 50 feet wide.

They erect a platform across the crater and drop rope at ten foot intervals across the platform to measure the depth of the crater at these locations. The first rope, ten feet in from the rim of the crater, is 40 feet long.



Placing this diagram on a coordinate system of your choice, find an equation for a parabolic arc that fits these initial data values. .

Assuming that the crater is indeed parabolic, what is the depth of the crater?

What, according to your equation, are the lengths of the remaining three ropes?

To test their conjecture that the crater is indeed parabolic, two more ropes of lengths 50.4 feet are to be hung from the platform. Predict exactly where along the platform they should be placed so as to just touch the floor of the crater.

PART III:
FITTING QUADRATICS TO DATA

FINDING QUADRACTICS WITH SPECIFIC ZEROS

Here's a question:

Write down a quadratic that crosses the x-axis at 2 *and at* 5.

Thinking for a moment one might suggest:

$$y = (x-2)(x-5)$$

This is a quadratic and we certainly have $y=0$ for $x=2$ and $x=5$.

Also, $y=3(x-2)(x-5)$ works, as does $y=-5(x-2)(x-5)$ and $y=-\frac{\sqrt{\pi}}{336+\sqrt{2}}(x-2)(x-5)$. In fact, any quadratic of the form $y=a(x-2)(x-5)$ does the trick.

EXERCISE:

a) Write down a quadratic with $x=677$ and $x=-6677$ as zeros.
b) Write down a quadratic that crosses the *x*-axis at 0 and at -3.
c) Write down a quadratic that touches the *x*-axis only at $x=40$.

Here's a slightly trickier question:

EXAMPLE: Write down a quadratic $y=ax^2+bx+c$ with *a*, *b* and *c* each an integer that crosses the *x*-axis at $x=\frac{4}{3}$ and at $x=\frac{50}{7}$.

Answer: Certainly $y=\left(x-\frac{4}{3}\right)\left(x-\frac{50}{7}\right)$ crosses the *x*-axis at these values but, when expanded, the coefficients involved won't be integers. To "counteract" the denominators of a 3 and a 7, what if we inserted the number 21 into this formula?

$$y=21\left(x-\frac{4}{3}\right)\left(x-\frac{50}{7}\right)$$

This is certainly still a quadratic with the desired zeros. And if we expand this slightly we see:

$$y = 3\cdot 7\cdot\left(x-\frac{4}{3}\right)\left(x-\frac{50}{7}\right)$$
$$y = 3\left(x-\frac{4}{3}\right)\cdot 7\left(x-\frac{50}{7}\right)$$
$$y = (3x-4)(7x-50)$$

When this is expanded fully, it is clear now that all the coefficients involved will be integers.

EXERCISE: Write examples of quadratics, involving only integers, with the following zeros:

a) $x=\frac{1}{2}$ and $x=\frac{1}{3}$

b) $x=-\frac{90}{13}$ and $x=\frac{19}{2}$

c) $x=5$ and $x=\frac{3}{7}$

d) Just one zero at $x=14\frac{3}{11}$

FITTING QUADRACTICS TO DATA

EXAMPLE: Find a quadratic function that fits the following data:

x	y
2	7
5	10
7	3

That is, find a quadratic function $y = ax^2 + bx + c$ that passes through the three points $(2,7)$, $(5,10)$ and $(7,3)$.

Answer: The best thing to do is to just write down the answer! Here it is:

$$y = 7 \cdot \frac{(x-5)(x-7)}{(-3)(-5)} + 10 \cdot \frac{(x-2)(x-7)}{3 \cdot (-2)} + 3\frac{(x-2)(x-5)}{5 \cdot 2}$$

If one were to expand this out we'd see that this is indeed a quadratic function. But, of course, this is not the issue in one's mind right now. Perhaps the question "From where does this formula come and what is it doing?" is more pressing!

To understand this meaty formula start by plugging in the value $x = 2$. (Do it!) Notice that the second and third terms are designed to vanish at $x = 2$ and so we have only to contend with the first term:

$$7 \cdot \frac{(x-5)(x-7)}{(-3)(-5)}$$

When $x = 2$ the numerator and the denominator match (the denominator was designed to do this) so that this term becomes:

$$7 \cdot 1$$

which is the value 7 we want from the table we were given.

For the value $x=5$ only the middle term

$$10\cdot\frac{(x-2)(x-7)}{3\cdot(-2)}$$

"survives" and has value $10\cdot\frac{3\cdot(-2)}{3\cdot(-2)}=10$ for $x=5$.

In the same way, for the value $x=7$ only the third term is non-vanishing and has value

$$3\frac{5\cdot 2}{5\cdot 2}=3$$

Thus the quadratic $y=7\cdot\frac{(x-5)(x-7)}{(-3)(-5)}+10\cdot\frac{(x-2)(x-7)}{3\cdot(-2)}+3\frac{(x-2)(x-5)}{5\cdot 2}$ does indeed produce the values 7, 10 and 3 for the inputs 2, 5, and 7!

COMMENT: If one so desired we can expand this to write:

$$y=-\frac{9}{10}x^2+\frac{219}{30}x-4$$

□

Despite the visual complication of the formula one can see that its construction is relatively straightforward:

1. Write a series of numerators that vanish at all but one of the desired inputs.
2. Create denominators that cancel the numerators when a specific input is entered.
3. Use the desired *y*-values as coefficients.

As another example, here's a quadratic that passes though the points $A=(3,87)$, $B=(10,\pi)$ and $C=(35,\sqrt{2})$:

$$y=87\frac{(x-10)(x-35)}{(-7)(-32)}+\pi\frac{(x-3)(x-35)}{(7)(-28)}+\sqrt{2}\frac{(x-3)(x-10)}{32\cdot 25}$$

CHECK: Put in $x=3$. Do you get the output 87? Also, put in $x=10$ and then $x=35$.

EXERCISE: Find a quadratic that fits the data

x	y
0	3
4	3
5	18

Simplify your answer.

EXAMPLE: Find a quadratic that fits the data

x	y
1	a
2	b
3	c

Answer: $y=a\frac{(x-2)(x-3)}{(-1)(-2)}+b\frac{(x-1)(x-3)}{1\cdot(-1)}+c\frac{(x-1)(x-2)}{2\cdot 1}$ □

EXERCISE: Find a quadratic that goes through the points $(-3,-14)$, $(2,1)$ and $(3,-2)$. Simplify your answer as much as possible.

EXERCISE: Something interesting happens if one tries to find a quadratic that fits the points $(2,7)$, $(3,9)$ and $(6,15)$.

a) Write down a quadratic that seems to fit these data points and simplify your answer.
b) What happened and why?

EXERCISE: Something goes wrong if one tries to find a quadratic that fits the data $(1,0)$, $(1,-2)$ and $(-1,-1)$.

a) Try to write a quadratic that fits this data.
b) What goes wrong and why?

CHALLENGE: Find an equation of the form $x = ay^2 + by + c$ that fits this data!